FUNDAÇÃO GETULIO VARGAS

EPGE

Escola de Pós-Graduação em Economia

Ensaios Econômicos

Escola de Pós-Graduação em Economia da Fundação Getulio Vargas

N° 478 ISSN 0104-8910

Temporal Aggregation and Bandwidth Selection in Estimating Long Memory

Leonardo Rocha Souza

Março de 2003

URL: http://hdl.handle.net/10438/874

Rocha Souza, Leonardo
Temporal Aggregation and Bandwidth Selection in Estimating Long Memory/ Leonardo Rocha Souza - Rio de Janeiro : FGV,EPGE, 2010
(Ensaios Econômicos; 478)

Inclui bibliografia.

CDD-330

Temporal Aggregation and Bandwidth Selection in Estimating Long Memory

Leonardo R. Souza
EPGE – Fundação Getúlio Vargas

June 2003

Abstract
This paper reinterprets results of Ohanissian et al (2003) to show the asymptotic equivalence of temporally aggregating series and using less bandwidth in estimating long memory by Geweke and Porter-Hudak's (1983) estimator, provided that the same number of periodogram ordinates is used in both cases. This equivalence is in the sense that their joint distribution is asymptotically normal with common mean and variance and unity correlation. Furthermore, I prove that the same applies to the estimator of Robinson (1995). Monte Carlo simulations show that this asymptotic equivalence is a good approximation in finite samples. Moreover, a real example with the daily US Dollar/French Franc exchange rate series is provided.

Keywords: Temporal Aggregation, Long Memory, Bandwidth, Spectrum.
JEL classification: C14, C22, C43

1 - Introduction

An important issue on long memory estimation is the level of temporal aggregation to apply to the time series in order to estimate the memory parameter. Crato and Ray (2002) explicitly advocate temporal aggregation of long memory time series with added noise in order to decrease the noise-to-signal ratio, whereas Ohanissian, Russell and Tsay (2003) propose temporal aggregation to distinguish between true and spurious long memory. Furthermore, while many authors have used different frequencies to estimate long memory in their empirical studies[1], other lot have studied the theoretical properties of temporally aggregated long memory processes[2]. Moreover, Monte Carlo simulations by Souza and Smith (2003) show that temporal aggregation may reduce the bias caused by short memory components while increasing the

[1] Diebold and Rudebusch (1989), Tschernig (1995), Bisaglia and Guégan (1998) and Chambers (1998), to name a few.
[2] For example, Tschernig (1995), Chambers (1998), Teles, Wei and Crato (1999) and Souza (2003).

estimates standard error, the latter conclusion apparently due only to the shortening of the series imposed by aggregation.

An older issue concerns the spectral bandwidth to use in semiparametric frequency-domain estimation methods for long memory. It is agreed that the wider bandwidth used, the lower the estimates standard error. On the other hand, as long memory relates to the low frequencies of the spectrum, using a larger bandwidth makes the semiparametric estimation more susceptible to biases due to short memory components (see, for example, Souza and Smith, 2002, Smith, Taylor and Yadav, 1997).

Reinterpreting results of Ohanissian, Russell and Tsay (2003), I show that, asymptotically (for fixed aggregation level), aggregating the series to estimate long memory by Geweke and Porter-Hudak's (1983) estimator (GPH) is equivalent to reducing the spectral bandwidth to a specific number (band) of frequencies, such that the same number of frequencies is used both in the original and in the aggregated series. This equivalence is in the sense that their joint distribution is asymptotically normal with common mean and variance and unity correlation. I prove the same equivalence for the Gaussian semiparametric estimator (GSPR) of Robinson (1995) and conjecture that some kind of equivalence must hold for other periodogram-based estimators. Monte Carlo simulations show that the correlation between aggregated and (specific) low bandwidth estimates approaches one very fast for ARFIMA (0,d,0) and ARFIMA (1,d,0) processes, but considerably slower if a negative moving average component is present. In addition, the estimates mean and standard error are very similar.

The daily US Dollar/ French Franc exchange rate series from October 20, 1977 to October 23, 2002 is studied. In a long memory stochastic volatility (Breidt, Crato and Lima, 1998) model framework, the logarithm of the squared returns are analysed and the absence of long memory is rejected by the Lo's (1991) modified R/S test. For different levels of aggregation and same number of frequencies used the variation in estimates is minimal compared to the same level of aggregation and different number of frequencies.

Section 2 briefly explains long memory processes and the GPH and GSPR estimators, as well as the asymptotic equivalence. Section 3 shows some numerical results, Section 4 studies the US Dollar/French Franc exchange rate series and Section 5 offers a final consideration. Technical details and proofs are relegated to the Appendix.

2 – Long memory processes

Stationary long memory processes are defined by the behaviour of the spectral density function near the frequency zero, as follows.

Definition 1: if there exists a positive function $c_f(\lambda)$, $\lambda \in (-\pi, \pi]$, which varies slowly as λ tends to zero, such that d ∈ (0, 0.5) and

$$f(\lambda) \sim c_f(\lambda)|\lambda|^{-2d} \text{ as } \lambda \to 0, \tag{1}$$

where f(λ) is the spectral density function of the stationary process X_t, then X_t is a stationary process with long memory with (long-)memory parameter d.

X_t is said to follow an ARFIMA(p,d,q) model if $\Phi(B)(1-B)^d X_t = \Theta(B)\varepsilon_t$, where ε_t is a mean-zero, constant variance white noise process, B is the backward shift operator such that $BX_t = X_{t-1}$, and $\Phi(B) = 1-\phi_1 B-\ldots-\phi_p B^p$ and $\Theta(B)=1+\theta_1 B+\ldots+\theta_q B^q$ are the short-run autoregressive and moving-average polynomials, respectively. ARFIMA processes are stationary and display long memory if the roots of Φ(B) are outside the unit circle and d ∈ (0, 0.5).

2.1 – The GPH estimator

The GPH estimator, proposed by Geweke and Porter-Hudak (1983), estimates d from the spectrum behaviour close to the zero frequency. Taking the log of the both sides of (1) yields $\log f(\lambda) \approx \log c_f(\lambda) - 2d \log \lambda$ in the positive vicinity of the zero frequency. Replacing the spectral density function by the periodogram $I(\lambda_j)$ and rearranging gives way to:

$$\log I(\lambda_j) = \log c_f(\lambda) - 2d \log \lambda_j + \xi_j, \tag{2}$$

where $\lambda_j = 2\pi j/T$, j = 1, …, m, are the Fourier frequencies and T is the sample size. Least-squares estimation applied to (2) yields an estimate for d. Hurvich, Deo and Brodsky (1998) prove that this estimator is consistent provided that the time series is Gaussian and $m \to \infty$ and $(m \log m)/T \to 0$ as $T \to \infty$. They also prove asymptotic normality:

$$\sqrt{m}\,(\hat{d} - d) \xrightarrow{D} N(0, \pi^2/24). \tag{3}$$

Note that the variance of the asymptotic distribution depends only on the number of Fourier frequencies used in the estimation. It is usual to consider m as a function of the series length (m = F(T)).

2.2 – The Gaussian semiparametric estimator of Robinson (GSPR)

This estimator was proposed by Robinson (1995) and maximises the approximate form of the frequency domain Gaussian likelihood, where discrete averaging is carried out over a neighbourhood of zero frequency:

$$R(d) = \log\left(\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d} I_j\right) - \frac{2d}{m}\sum_{j=1}^{m}\log(\lambda_j)\,. \qquad (4)$$

Robinson (1995) outlines the conditions under which this estimator is consistent and the ones under which it is asymptotically Gaussian so that:

$$\sqrt{m}\,(\hat{d} - d) \xrightarrow{D} N(0,¼)\,. \qquad (5)$$

It is important to point out that (5) is proved without imposing Gaussianity in the series. Again, the asymptotic variance depends only on the number of periodogram ordinates used in the estimation, but note that the GSPR has lower asymptotic variance than the GPH if we consider the same number m of periodogram frequencies used. However, one must bear in mind that different assumptions are made in proving the results for the two estimators, details of which are found in the respective papers.

2.3 – Temporal aggregation of long memory processes

If one considers n as the level of temporal aggregation, it is equivalent to observing a flow variable at a frequency 1/n times the original one. In other words, summing up every n-th and its preceding n-1 observations. The aggregated variable Y_t is observed as follows:

Definition 2: Let X_t be a process observed at times t = 1, …, T_X. Then $Y_t = \sum_{i=0}^{n-1} X_{nt-i} = \sum_{i=0}^{n-1} B^i X_{nt}$, t = 1, …, T_y ; $T_y = T_X/n$.

Ohanissian, Russell and Tsay (2003) prove that if the series is Gaussian then the GPH estimators from the original and the aggregated series are asymptotically jointly normal, and the covariance between them is asymptotically equal to the variance of the estimate from the original series. Although they consider that less frequencies are used in the estimation of the aggregated series than in the estimation of the original series ($m_X > m_Y$), the proof is still valid and hence the result holds for the cases in which the same number of frequencies is used for both. In this case ($m_X = m_Y$), as asymptotically the estimator variance depends only on the number of frequencies used (see equation (3)), the variances are equal and the correlation approaches one as $T \to \infty$.

To sum up, these estimates are asymptotically unbiased, jointly Gaussian with the same variance and correlation one. Reinterpreting their results, there is an asymptotic equivalence between temporally aggregating a variable in order to estimate long memory by the GPH method and using a specific lower number of frequencies to estimate the long memory in the original series, such that the same number of frequencies is used in the original and in the aggregated series. Consider, for example, the case where $m = F(T) = T^{\alpha}$, $0<\alpha<1$. Asymptotically, estimating X_t using $F(T_X)$ frequencies is equivalent to estimate Y_t using $F^*(T_Y) = F(T_X) = n^{\alpha}F(T_Y)$. Or, alternatively, estimating X_t and Y_t both using $F(T_Y)$ frequencies yields asymptotically equivalent estimates.

In this paper I prove that the same equivalence applies to the GSPR estimator. However, it is not assumed Gaussianity in the data as in the GPH. Furthermore, Lemma 1 of Ohanissian et al. (2003) allows one to conjecture that other semiparametric long memory estimators must have some kind of asymptotic equivalence. The technical proof and details are found in the Appendix. The next Section contains simulations that show that even with a small sample size the estimates from the original and the aggregated series are correlated almost to the unity, both for the GPH and the GSPR.

3 – Simulations

This Section presents the results of simulations with Gaussian ARFIMA series. The simulation exercise consists of generating synthetic series of different lengths (T_X = 200, 500, 1000) and computing mean, standard deviation and correlations between the estimates from the original series and aggregates up to aggregation level equal to 6 ($n = 2, \ldots, 6$) over 500 replications of each model. The number of periodogram ordinates (m) to be used in the estimation, however, is held fixed across all aggregation levels, even though the number of observations decreases from T_X (original series) to $T_{Y,n} = T_X/n$ (aggregated series with aggregation level n). This is equivalent to use a shorter bandwidth in the long memory estimation of the original series. The idea is to compare the estimation with “reduced bandwidth” on the original series to the one with “usual bandwidth” on the aggregated series, illustrating thus in the finite sample the asymptotic equivalence between aggregating the series and reducing the bandwidth. To be precise, m should vary with $T_{Y,n}$ instead of T_X (how the experiment is designed). However, both cases are equivalent once m is the same in estimating X_t and Y_t, and we use for simplicity the original series sample size as the argument of F(T). Doing so, there is no need to estimate the long memory of X_t for every $m = F(T_{Y,n})$, $n = 2, \ldots, 6$, but only for $m = F(T_X)$. Given the asymptotic joint Gaussianity, these statistics are sufficient to specify the distribution of estimates for large samples.

The models considered are ARFIMA (0,d,0), ARFIMA (1,d,0) with ϕ = 0.8, and ARFIMA (0,d,1) with θ = -0.8, for d = -0.3, -0.1, 0, 0.1, 0.3. Table 1 shows mean and standard deviation of GPH estimates of ARFIMA (0,d,0) aggregates up to n = 6 (the original series is equivalent to an aggregate with n = 1) and $m = T_X^{\alpha}$, α = 0.4, 0.5 and 0.6, for all the series, where T_X is the length of the original series. The mean and the standard deviation of the estimates must be compared across lines, as the aggregates are disposed in columns. Note that when T_X = 200, there are not enough periodogram observations to compute the GPH estimates for n = 5, 6 and m = $T_X^{0.6}$. As to the results, the standard deviation remains almost constant across aggregates and varies slightly across values of d, this variation probably due to sample variation (which does not occur across aggregates). In this small sample exercise, the estimates standard deviation is apparently only determined by m, as in the asymptotic behaviour. However, they are higher than their asymptotic counterparts, given as follows; 0.227, 0.185 and 0.166 respectively for T_X = 200, 500 and 1000 and α = 0.4; 0.171, 0.137 and 0.115 for α = 0.5; and 0.131, 0.100 and 0.081 for α = 0.6. The bias is small but increases slightly as the series is aggregated, increasing the absolute value of the estimates. The larger the bandwidth used the more marked are the differences between estimates. Table 2 shows the corresponding results for the GSPR. They are qualitatively similar to those of the GPH estimator, attaining, however, lower standard deviation for all processes. The bias is comparable. Moreover, the results are slightly more uniform than for the GPH, both for the mean and the standard deviation.

It is well known that first-order negative AR and positive MA components do not entail substantial bias in long memory estimation. The corresponding results are not shown but are available from the author under request. A positive AR and a negative MA components, however, bias respectively upward and downward long memory estimation (see, for example, Souza and Smith (2002) and Smith, Taylor and Yadav (1997)). This is what is observed in Table 3. Table 3 shows the results for the GPH estimates for ARFIMA (1,d,0), ϕ = 0.8, and ARFIMA (0,d,1), θ = -0.8, processes, with α = 0.5. The results concerning standard deviation of the estimates agree with those from Table 1 (α = 0.5), being slightly larger in some cases, especially when the process is an ARFIMA(0,-0.3,1). The bias increases in some cases heavily with aggregation, particularly when d = -0.3. Using the same number of periodogram ordinates as in the original series offsets the bias reduction obtained by aggregating series in the presence of short memory components (see Souza and Smith, 2003), while keeping constant the variance of the estimates. In general, the MA increases the gap between the average estimates from the original and the aggregated series, whereas the effects of the AR component in this gap in not clear, sometimes increasing and other times decreasing it. Table 4 shows the same as Table 3, but for the GSPR instead of the GPH. The results are qualitatively similar to those from the GPH and the bias is comparable across all processes. The standard deviation, however, is lower for the GSPR and varies slightly less across aggregates.

Table 5 shows the correlations between GPH estimates from the original series and the aggregates up to order 6. The results refer to the same processes and bandwidths considered in Tables 1 and 3. The correlations are very high, being virtually one in some cases (specially between the original series and n = 2, positive values of d, highest sample sizes and when an AR is present). Regarding the results from previous tables and this one, we conclude that in general the asymptotic equivalence is a good approximation in small samples. The correlation increases with the series length, as expected, and with d for all processes studied and bandwidths tried. The increase with d is expected, at least for the GSPR, as the convergence order depends on the value of this parameter, increasing as d increases (details in the Appendix). On the other hand, it decreases as the aggregation level n increases. It is noticed that the less bandwidth is used the closer to unity are the correlations. Remember that in Table 1 the difference between means are less marked in this case. These results allow us to conclude that the larger the bandwidth used in the estimation (the higher m) the larger must be T to the asymptotic relationship be a good approximation to the actual relationship. Furthermore, these results make sense with the theory, since the order of convergence (of the asymptotic equivalence) is dependent on the number of periodogram ordinates used in the estimation[3]. Adding short memory components to the purely fractionally integrated process affects the results as follows: the AR component seems to accentuate the correlation, whereas the MA inflicts the inverse consequence. Remember the effects of these components on the gap between average estimates from the original and the aggregated series. For small samples, hence, the equivalence is a better approximation when an AR component is present and is a worse one when there is an MA component, observed the parameter signs and magnitude used. Table 6 shows the same as Table 5, but for the GSPR estimation method. This method yields correlations consistently higher than those from the GPH, albeit by a small margin, hinting that the equivalence as an approximation in small samples is more accurate for the GSPR. The results for the GSPR are consistent with those for the GPH.

If instead of $m = F(T_X)$ one considers $m = F(T_X/n)$, which is the usual way to proceed in practice, the correlations are high but not so close to the unity. For example, for the ARFIMA

[3] The narrower bandwidth the faster is the convergence between estimates from the original and the aggregated series. However, in this case the estimates converge slower to the true parameter.

(0,d,0), the correlation between GPH estimates with n = 1 and n = 2 are around ¾ for all values of d considered and T = 200; and around 0.8 for all values of d considered and T = 1000. These correlations, although of interest, are less relevant to the paper results. They are not shown in tables here but are available from the author under request, as well as the corresponding correlations not shown here for some combinations of processes and bandwidths.

4 – Real example

This example aims to verify in an actual series what one should expect the proximity between estimates would be if long memory was estimated from aggregates with different levels using the same number of periodogram ordinates. For this purpose, the daily US Dollar/ French Franc (US$/FF) exchange rate series is considered from October 20, 1977 to October 23, 2002 (25 years). More specifically, the natural logarithm of the squared returns is analysed. There are 68 (approximately 1.09%) zero returns existent in the 6264 workdays which were simply skipped, as well as the holidays. The series, its autocorrelation function (ACF) up to lag 1000 and its periodogram are shown in Figure 1, where the reader can notice the apparent long memory features such as persistently positive ACF (up to lag 250), and the periodogram scattered around a frequency power near the frequency zero. The same series is studied in Souza (2003) and is consistent with the Long Memory Stochastic Volatility (LMSV) model proposed by Breidt, Crato and Lima (1998), which is given by the following relation:

$$R_t = \sigma \exp(Y_t / 2)\varepsilon_t , \qquad (6)$$

where Y_t is a stationary Gaussian long memory process independent of ε_t, mean zero iid white noise, and R_t is the (log-)return. The analysed series is then:

$$Z_t \equiv \log(R_t^2) = \mu + Y_t + v_t , \qquad (7)$$

where $\mu = (\log \sigma^2 + E[\log \varepsilon_t^2])$ and $v_t = (\log \varepsilon^2 - E[\log \varepsilon_t^2])$ is iid mean zero. Z_t is then a sum of a Gaussian long memory process and a white noise. The kurtosis of the series in study is approximately 3.68 and the skewness –0.79, so that the Jarque-Bera test rejects the hypothesis of Gaussianity at 1% confidence level. This does not mean that the Gaussianity of the non-observable Y_t is rejected since it is contaminated by the noise v_t in the observed Z_t. Furthermore, the reportedly conservative (Teverovsky et al., 1999) modified R/S test of Lo (1991) rejects the hypothesis of short memory in Z_t at a 0.5% level. Although the series is of stock type, aggregating it as flow is advocated by Crato and Ray (2002) in order to decrease bias from estimating long memory from a series with added noise as Z_t.

Table 7 shows GPH and GSPR estimates of long memory for aggregates from aggregation levels up to 10 (lines), and different number of periodogram observations used in the estimation (columns; m = 10, 20, 30, 40, 60, 80, 100, 120, 150). It is apparent that the variation within column (different levels of aggregation, same m) is minimal compared to the variation across columns (same aggregation levels, different m's) and this is more pronounced for GSPR. This illustrates the asymptotic equivalence between aggregating the series and using a narrower bandwidth in the original one provided that one holds fixed the number of periodogram observations to be used in the estimation. In other words, there is no need to aggregate the series just to diminish the bias, it is enough to use a narrower bandwidth in the estimation.

5 – Final considerations

There are two related discussions concerning the long memory estimation in time series. One is about the trade-off implied by aggregating the series before semiparametric estimation and the other concerns the bandwidth to use in semiparametric frequency-domain estimation methods. Aggregating, as well as using less bandwidth, is known to reduce the bias induced by short memory components while increasing the estimates standard error.

This paper shows, based on results from Hurvich et al (1998), Ohanissian et al (2003) and Robinson (1995), that, for long memory estimation purposes, aggregating is asymptotically equivalent to use a specific lower bandwidth, holding fixed the aggregation level. This specific lower bandwidth is such that the same number of periodogram ordinates is used in the original and the aggregated series. The results are valid for the Geweke and Porter-Hudak (1983) and the GSPR (Robinson, 1995) estimators, but some kind of equivalence must hold for other periodogram-based semiparametric long memory estimators. A small simulation is provided to show that, in addition to the estimates mean and standard deviation being very similar, the correlation between estimates is close to unity even for small sample sizes. These results, however, are affected by other factors than the sample size, such as the memory parameter d, the aggregation level, the presence of a short memory component and the bandwidth used in the estimation. An additional example with the US$/FF exchange rate series illustrates that aggregating the series makes little difference when using the same number of periodogram ordinates in the estimation.

Acknowledgements

The author would like to thank FAPERJ for the financial support.

References

Bisaglia, L. and Guégan, D. (1998), "A comparison of techniques of estimation in long-memory processes", *Computational Statistics & Data Analysis*, 27, 61-81.

Breidt, F.J., Crato, N. and Lima, P. (1998), "The detection and estimation of long memory in stochastic volatility", *Journal of Econometrics*, 83, 325-348.

Chambers, M. J. (1998), "Long memory and aggregation in macroeconomic time series", *International Economic Review*, 39, 1053-1072.

Crato, N. and Ray, B. K. (2002), "Semi-parametric smoothing estimators for long-memory processes with noise", *Journal of Statistical Planning and Inference*, 105, 283-97.

Diebold, F. X. and Rudebusch, G. D. (1989), "Long memory and persistence in aggregate output", *Journal of Monetary Economics*, 24, 189-209.

Geweke, J. and Porter-Hudak, S. (1983), "The estimation and application of long memory time series models", *Journal of Time Series Analysis*, 4, 221-237.

Hosking, J.R.M. (1996), "Asymptotic distributions of the sample mean, autocovariances, and autocorrelations of long-memory time series", *Journal of Econometrics* 73, 261-284.

Hurvich, C. M., Deo, R. and Brodsky, J. (1998), "The mean square error of Geweke and Porter-Hudak's estimator of the memory parameter of a long-memory time series", *Journal of Time Series Analysis*, 19, 19-46.

Lo, A.W. (1991), "Long-term memory in stock market prices", *Econometrica*, 59, 5, 1279-1313.

Ohanissian, A., Russell, J. and Tsay, R. (2003), "Using temporal aggregation to distinguish between true and spurious long memory", *working paper*, available at http://gsb-www.uchicago.edu/fac/jeffrey.russell/research/wp.htm.

Robinson, P. M. (1995), "Gaussian semi-parametric estimation of long range dependence", *Annals of Statistics*, 23, 5, 1630-1661.

Smith, J., Taylor, N. and Yadav, S. (1997), "Comparing the Bias and Misspecification in ARFIMA Models", *Journal of Time Series Analysis*, 18, 507-528.

Souza, L.R. (2003), "The aliasing effect, the Fejer kernel and temporally aggregated long memory processes", *working paper*, available at http://www.fgv.br/epge/home/PisDownload/1086.pdf.

Souza, L. R. and Smith, J. (2002), "Bias in the memory parameter for different sampling rates", *International Journal of Forecasting*, 18, 299-313.

Souza, L. R. and Smith, J. (2003), "Effects of temporal aggregation on estimates and forecasts of fractionally integrated processes: A Monte Carlo study", *International Journal of Forecasting*, forthcoming.

Teles, P., Wei, W. W. S., and Crato, N. (1999), "The use of aggregate series in testing for long memory", *Bulletin of the International Statistical Institute*, 52nd Session Book 3, 341-342.

Teverovsky, V., Taqqu, M.S. and Willinger, W. (1999), "A critical look at Lo's modified R/S statistic", *Journal of Statistical Planning and Inference*, 80, 211-27.

Tschernig, R. (1995), "Long memory in foreign exchange rates revisited", *Journal of International Financial Markets, Institutions and Money*, 5, 53-78.

Appendix

- **Proof of the asymptotic joint normality with unity correlation between GSPR estimates from the original and the aggregated series using the same number of periodogram ordinates m.**

Let I^X_j and I^Y_j be the periodogram ordinates for the j-th Fourier frequency of the original and the aggregated series respectively. Lemma 1 of Ohanissian et al (2003) reads as $I_j^Y \xrightarrow{P} nI_j^X$ as $T \to \infty$, provided that: X_t is Gaussian; a condition similar to Definition 1 holds; and the level of aggregation n increases at a slower rate than T. However, their proof seems to be missing a result, so that I provide a revision on it for n = 2, which is found further in the Appendix. The revised proof makes an alternative assumption on the long memory of the series, imposing it in the time-domain rather than in the frequency domain. Furthermore, the revision can be directly applied to their proof with general level of aggregation n, as long it is held fixed. This result refers to a convergence in probability, and the revision upon their proof provided in this paper allows one to reach a convergence speed different than theirs:

$$I_j^Y = nI_j^X + O(j^2/T) + O_P(j/T^{1-2d}) \text{ as } T \to \infty. \tag{A1}$$

If j is fixed, the second term on the RHS is $O(T^{-1})$ and the third is $O_P(T^{2d-1})$, but one must remember that j varies from 1 to m in the estimation methods, and m is related to T. Thus, one must consider these terms as $O(m^2/T)$ and $O_P(m/T^{1-2d})$, respectively. As to the convergence speed in the general case, note that if $I_j^Y = 2I_j^X + O(j^2/T) + O_P(j/T^{1-2d})$ as $T \to \infty$ for n = 2, then (A1) holds at least for $n = 2^k$, k = 1, 2, 3, …, easily verified by induction.

Assumption 1A: $T^{-1}m^{2d+3/2}(1+T^{-2d}m)\log(T/m) \to 0$ as $T \to \infty$.

This assumption may exclude some values of (m, d) used in the simulation. However, it is still worthwhile keeping them in the simulation, since only the sufficiency of Assumption 1A, together with Assumptions A1' – A4' of Robinson was proved. Moreover, those values of (m, d) are kept, if not for other reason, for illustration purposes.

Let also Assumptions A1' – A4' of Robinson (1995) hold. Robinson proves (5) so that $\sqrt{m}\,(\hat{d}-d)\xrightarrow{D} N(0,¼)$. Let us consider $\hat{d}_n$ the estimator from the aggregated series using the same number m of periodogram ordinates as the estimator from the original series $\hat{d}$. So, (5) also holds for $\hat{d}_n$, such that $\hat{d}$ and $\hat{d}_n$ have the same asymptotic distribution (and hence the same asymptotic mean and variance). If we prove that $\hat{d}_n \to_P \hat{d}$ as T $\to \infty$ faster than $\hat{d} \to_P d$, then the proof is complete, as in this case $am^{1/2}\hat{d}_n + bm^{1/2}\hat{d} = (a+b)m^{1/2}\hat{d} + o_P(1)$ as T $\to \infty$, for any constants a, b. That means that any linear combination of $\hat{d}$ and $\hat{d}_n$ is asymptotic normal and they are asymptotically unity correlated. For this, we have to prove that $\hat{d}_n = \hat{d} + o_P(m^{-1/2})$ asymptotically. For ease of comparison with Robinson formulas, let H_0 = d + 1/2; $\hat{H} = \hat{d} + 1/2$; and $\hat{H}_n = \hat{d}_n + 1/2$. For those reading the work of Robinson (1995), please note, however, that n in his notation is equivalent to T in ours and refers to the sample size. Equation (4.2) of Robinson (1995) states that with probability approaching 1, as T $\to \infty$,

$0 = \dfrac{dR(H_0)}{dH} + \dfrac{d^2R(\tilde{H})}{dH^2}\left(\hat{H}-H_0\right)$, where $\left|\tilde{H}-H_0\right| \le \left|\hat{H}-H_0\right|$ and R(H) is as defined in (4), but replacing d by (H – ½). Rewriting this, we have:

$$\hat{H}-H_0 = \frac{-dR(H_0)\big/dH}{d^2R(\tilde{H})\big/dH^2}. \qquad \text{(A2)}$$

Robinson (1995) states further that

$$\frac{dR(H)}{dH} = 2\frac{\hat{G}_1(H)}{\hat{G}_0(H)} - \frac{2}{m}\sum_{j=1}^{m}\log\lambda_j \qquad \text{(A3)}$$

and

$$\frac{d^2R(H)}{dH^2} = 4\frac{\left\{\hat{G}_2(H)\hat{G}_0(H)-\hat{G}_1^2(H)\right\}}{\hat{G}_0^2(H)}, \qquad \text{(A4)}$$

where

$$\hat{G}_k(H) = \frac{1}{m}\sum_{j=1}^{m}\left(\log\lambda_j\right)^k\lambda_j^{2H-1}I_j\,.$$

Note that $\lambda_j = \frac{2\pi j}{T}$, so it is O(m/T) as $\lambda \to 0+$. Furthermore, $\lambda_j^{2H-1} I_j = O_P(1)$ as $\lambda \to 0+$ if H = $H_0 + O_P(m^{-1/2})$. Hence $\hat{G}_k(H) = O_P(\log^k(m/T)) = O_P(\log^k(T/m))$.

Now we consider the estimate $\hat{H}_n$ from the aggregated series Y_t. Let $\hat{G}_{k,n}(H)$ and $R_n(H)$ denote for Y_t the equivalent to $\hat{G}_k(H)$ and R(H) for the original series X_t, holding m fixed across aggregates. Note that the Fourier frequencies λ^Y_j of Y_t are related to the frequencies λ_j of X_t by $\lambda^Y_j = n\lambda_j$. Using (1) – that is somewhat similar to, though less formally stated than, Assumption A1' of Robinson (1995) – and (A1), we have then:

$$\hat{G}_{k,n}(H) = \frac{1}{m}\sum_{j=1}^{m}\left(\log n\lambda_j\right)^k n^{2H-1}\lambda_j^{2H-1}(nI_j^X + O(m^2/T) + O_P(m/T^{2-2H_0})) =$$

$$= \frac{n^{2H}}{m}\sum_{j=1}^{m}\left(\log n + \log\lambda_j\right)^k \lambda_j^{2H-1}(I_j^X + O(m^2/T) + O_P(m/T^{2-2H_0})) = \tag{A5}$$

$$= \frac{n^{2H}}{m}\sum_{j=1}^{m}\left(\log n + \log\lambda_j\right)^k \lambda_j^{2H-1} I_j^X + O\left(\frac{m^{1+2H}}{T^{2H}}\log^k\frac{m}{T}\right) + O_P\left(\frac{m^{2H}}{T^{1+2\Delta H}}\log^k\frac{m}{T}\right),$$

where $\Delta H = H - H_0$. A particularization of (A5) for k = 0, 1, 2, followed by straightforward calculation, gives:

$$\hat{G}_{0,n}(H) = n^{2H}\hat{G}_0(H) + O\left(\frac{m^{1+2H}}{T^{2H}}\right) + O_P\left(\frac{m^{2H}}{T^{1+2\Delta H}}\right);$$

$$\hat{G}_{1,n}(H) = n^{2H}\left\{\hat{G}_1(H) + \log n\hat{G}_0(H)\right\} + O\left(\frac{m^{1+2H}}{T^{2H}}\log\frac{T}{m}\right) + O_P\left(\frac{m^{2H}}{T^{1+2\Delta H}}\log\frac{T}{m}\right); \tag{A6}$$

$$\hat{G}_{2,n}(H) = n^{2H}\left\{\hat{G}_2(H) + 2\log n\hat{G}_1(H) + \log^2 n\hat{G}_0(H)\right\} + O\left(\frac{m^{1+2H}}{T^{2H}}\log^2\frac{T}{m}\right) + O_P\left(\frac{m^{2H}}{T^{1+2\Delta H}}\log^2\frac{T}{m}\right)$$

Note that the second term on the right hand side of the equations (A6) dominates the third when H < ½, and the inverse occurs when H > ½. Adapting (A2) for $\hat{H}_n$ gives:

$$\hat{H}_n - H_0 = \frac{-dR_n(H_0)/dH}{d^2R_n(\tilde{H})/dH^2}, \tag{A7}$$

where

$$\frac{dR_n(H)}{dH} = 2\frac{\hat{G}_{1,n}(H)}{\hat{G}_{0,n}(H)} - \frac{2}{m}\sum_{j=1}^{m}\log n\lambda_j \tag{A8}$$

and

$$\frac{d^2 R_n(H)}{dH^2} = 4\frac{\left\{\hat{G}_{2,n}(H)\hat{G}_{0,n}(H) - \hat{G}_{1,n}^2(H)\right\}}{\hat{G}_{0,n}^2(H)}. \tag{A9}$$

Remembering that $\hat{G}_k(H) = O_P(\log^k(T/m))$, further calculation on (A8) and (A9), using (A6), gives:

$$\frac{dR_n(H_0)}{dH} = \frac{dR(H_0)}{dH} + O_P\left(\frac{m^{1+2H_0}}{T^{2H_0}}\log(T/m)\right) + O_P\left(\frac{m^{2H_0}}{T}\log(T/m)\right) \tag{A10}$$

and

$$\frac{d^2 R_n(\tilde{H})}{dH^2} = \frac{d^2 R(\tilde{H})}{dH^2} + O_P\left(\frac{m^{1+2\tilde{H}}}{T^{2\tilde{H}}}\log^2(T/m)\right) + O_P\left(\frac{m^{2\tilde{H}}}{T^{1+2(\tilde{H}-H_0)}}\log^2(T/m)\right). \tag{A11}$$

Equations (4.10) and (4.11) of Robinson (1995) and extensions show that $\frac{dR(H_0)}{dH} = O_P(m^{-1/2})$ and $\frac{d^2 R(\tilde{H})}{dH^2} = O_P(1)$. So the only additional requirement (to A1'–A4' of Robinson(1995)) to $\hat{H}_n$ converge to $\hat{H}$ faster than $\hat{H}$ to H_0 is that $\left(\frac{m^{1+2H_0}}{T^{2H_0}} + \frac{m^{2H_0}}{T}\right)\log(T/m) = o\left(m^{-1/2}\right)$. This is satisfied by the Assumption 1A. The proof is complete.

- **Proof of (A1) with n = 2**

This proof is a revision on a specific case of Lemma 1 of Ohanissian et al. (2003). This revision, however, is directly applicable to a more general case, where n > 2 but is held fixed. The proof will require an alternative assumption (1B), which is given below. Also, the Proposition 1B is provided below in order to make the proof simpler. Note that j is to be considered as O(m), since j = 1, …, m.

Assumption 1B: X_t is stationary and there exists a real number d < 0.5 and a positive function $c_\rho(k)$ which varies slowly as k tends to infinity, such that $\rho_X(k) \sim c_\rho(k)k^{2d-1}$ as $k \to \infty$, where $\rho_X(k)$ is the k-th order autocorrelation of X_t.

Proposition 1B: $\cos\left(\frac{2\pi j}{T}(2t_1 \pm x)\right) = \cos\left(\frac{2\pi j}{T}2t_1\right) \mp \frac{2\pi jx}{T} sin\left(\frac{2\pi j}{T}2t_1\right) + O\left(\left(\frac{jx}{T}\right)^2\right)$ as $T \to \infty$.

Proof: Proposition 1B is easily verified by expanding $\cos\left(\frac{2\pi j}{T}(2t_1 \pm x)\right)$ by a Taylor series around $\cos\left(\frac{2\pi j}{T}(2t_1)\right)$. CQD.

Take the periodogram of X_t at the j-th Fourier frequency:

$$I_j^X = \frac{1}{2\pi T}\left[\sum_{t=1}^{T} X_t^2 + 2\sum_{t_1=1}^{T}\sum_{t_2=t_1+1}^{T} X_{t_1}X_{t_2}\cos\left(\frac{2\pi j}{T}(t_2 - t_1)\right)\right] \tag{B1}$$

Let n = 2. Then the periodogram of Y_t at the j-th Fourier frequency, as a function of X_t, is as follows:

$$I_j^Y = \frac{1}{\pi T}\left[\sum_{t=1}^{T} X_t^2 + \sum_{t=1}^{T/2} X_{2t}X_{2t-1} + 2\sum_{t_1=1}^{T/2-1}\sum_{t_2=t_1+1}^{T/2}(X_{2t_1} + X_{2t_1-1})(X_{2t_2} + X_{2t_2-1})\cos\left(\frac{4\pi j}{T}(t_2 - t_1)\right)\right] \tag{B2}$$

After some manipulation on (B1) and (B2), it is shown that I_j^X and I_j^Y relate to each other as follows: $I_j^Y = 2I_j^X + \mathrm{A} + \mathrm{B}$, where

$$\mathrm{A} = \frac{2\pi}{T}\left\{\sum_{t=1}^{T/2} X_{2t}X_{2t-1}\left(1 - \cos\frac{2\pi j}{T}\right)\right\} \text{ and} \tag{B3}$$

$$\mathrm{B} = \frac{2\pi}{T}\sum_{t_1=1}^{T/2-1}\sum_{t_2=1}^{T/2-t_1}\left(\begin{array}{l} X_{2t_2}X_{2t_2+2t_1-1}\left[\cos\left(\frac{2\pi j}{T}(2t_1)\right) - \cos\left(\frac{2\pi j}{T}(2t_1 - 1)\right)\right] + \\ + X_{2t_2-1}X_{2t_2+2t_1}\left[\cos\left(\frac{2\pi j}{T}(2t_1)\right) - \cos\left(\frac{2\pi j}{T}(2t_1 + 1)\right)\right]\end{array}\right). \tag{B4}$$

A is $O_P((j/T)^2)$, since

$$1 - \cos\frac{2\pi j}{T} = \left[1 - \left(1 - \frac{1}{2!}\left(\frac{2\pi j}{T}\right)^2 + \frac{1}{4!}\left(\frac{2\pi j}{T}\right)^4 - \ldots\right)\right] = O\left(\left(\frac{j}{T}\right)^2\right).$$

The revision upon their proof actually occurs from now forth. In order to match the lag between variables being multiplied in B, one can take the first term relating to t_1=1 and the second term relating to t_1=T/2 – 1 out from the summation on t_1. Thus B rewrites as follows:

$$B = \frac{2\pi}{T}\left\{\sum_{t_2=1}^{T/2-1} X_{2t_2} X_{2t_2-1}\left[\cos\frac{4\pi j}{T} - \cos\frac{2\pi j}{T}\right] + \sum_{t_2=1}^{1} X_{2t_2-1} X_{2t_2+T-2}\left[\cos\frac{2\pi j}{T}(T-2) - \cos\frac{2\pi j}{T}(T-1)\right]\right\} + E$$

where $$E = \frac{2\pi}{T}\sum_{t_1=1}^{T/2-2}\sum_{t_2=1}^{T/2-t_1}\left(\begin{array}{l} X_{2t_2} X_{2t_2+2t_1+1}\left[\cos\left(\frac{2\pi j}{T}(2t_1+2)\right) - \cos\left(\frac{2\pi j}{T}(2t_1+1)\right)\right] + \\ + X_{2t_2-1} X_{2t_2+2t_1}\left[\cos\left(\frac{2\pi j}{T}(2t_1)\right) - \cos\left(\frac{2\pi j}{T}(2t_1+1)\right)\right] \end{array}\right). \qquad \text{(B5)}$$

Using Proposition 1B, it is straightforward to verify that cos(4πj/T) – cos(2πj/T) and cos[(T-2)2πj/T] – cos[(T-1)2πj/T] are both O(j/T). Thus, B = $O_P(j/T) + O_P(j/T^2)$ + E.

Now note that

$$\frac{2}{T}\sum_{t_2}^{T/2-t_1} X_{2t_2} X_{2t_2+2t_1+1} = \tilde{\gamma}^{X}_{2t_1+1} + O_P(n^{2d-1}) \quad \text{as } T \to \infty \text{ and}$$

$$\frac{2}{T}\sum_{t_2}^{T/2-t_1} X_{2t_2-1} X_{2t_2+2t_1} = \tilde{\gamma}^{X}_{2t_1+1} + O_P(n^{2d-1}) \quad \text{as } T \to \infty,$$

where $\tilde{\gamma}^{X}_{k} = \gamma^{X}_{k} + \mu^2$; μ being the unconditional mean of X_t (details in Hosking, 1996, p.266), since Assumption 1B holds (a sufficient condition). Hence, and using Proposition 1B and (B5), E can be written as:

$$E = \pi\sum_{t_1=1}^{T/2-2}\left(\begin{array}{l} (\tilde{\gamma}^{X}_{2t_1+1} + O_P(T^{2d-1}))\left[-\frac{2\pi j}{T} sin\left(\frac{2\pi j}{T}(2t_1+1)\right) + O\left(\frac{j^2}{T^2}\right)\right] + \\ + (\tilde{\gamma}^{X}_{2t_1+1} + O_P(T^{2d-1}))\left[\frac{2\pi j}{T} sin\left(\frac{2\pi j}{T}(2t_1+1)\right) + O\left(\frac{j^2}{T^2}\right)\right] \end{array}\right) \qquad \text{(B6)}$$

Cancelling terms, we have from (B6)

$$E = O\left(\frac{j^2}{T}\right) + \pi\sum_{t_1=1}^{T/2-2}\left(\begin{array}{l} O_P(T^{2d-1})\left[-\frac{2\pi j}{T} sin\left(\frac{2\pi j}{T}(2t_1+1)\right) + O\left(\frac{j^2}{T^2}\right)\right] + \\ + O_P(T^{2d-1})\left[\frac{2\pi j}{T} sin\left(\frac{2\pi j}{T}(2t_1+1)\right) + O\left(\frac{j^2}{T^2}\right)\right] \end{array}\right) \qquad \text{(B7)}$$

Hence $E = O\left(\frac{j^2}{T}\right) + O_P\left(\frac{j^2}{T^{2-2d}}\right) + O_P\left(\frac{j}{T^{1-2d}}\right) = O\left(\frac{j^2}{T}\right) + O_P\left(\frac{j}{T^{1-2d}}\right)$ and (A1) holds for n = 2 and, by induction, for n = 2^k, where k is an integer greater than zero (the proof for k = 0 is trivial, since the aggregated series is by definition equal to the original one). CQD.

Table 1: Mean and standard deviation of GPH estimates from various levels of aggregation, holding fixed the number of periodogram ordinates for ARFIMA(0,d,0). $G(T) = T^{\alpha}$, α = 0.4, 0.5 and 0.6.

	T	200						500						1000					
	d \ n	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
α = 0.4	-0.3	-0.274	-0.291	-0.285	-0.312	-0.323	-0.333	-0.299	-0.304	-0.305	-0.317	-0.322	-0.326	-0.289	-0.289	-0.294	-0.296	-0.305	-0.299
(0,d,0)		(0.340)	(0.335)	(0.329)	(0.350)	(0.357)	(0.352)	(0.244)	(0.243)	(0.253)	(0.251)	(0.256)	(0.252)	(0.220)	(0.219)	(0.232)	(0.225)	(0.215)	(0.224)
	-0.1	-0.108	-0.111	-0.104	-0.111	-0.129	-0.132	-0.088	-0.089	-0.088	-0.093	-0.092	-0.095	-0.096	-0.098	-0.094	-0.098	-0.095	-0.093
		(0.368)	(0.366)	(0.369)	(0.374)	(0.386)	(0.373)	(0.263)	(0.263)	(0.268)	(0.267)	(0.265)	(0.261)	(0.211)	(0.212)	(0.217)	(0.212)	(0.210)	(0.222)
	0	0.004	0.001	-0.001	0.006	0.003	-0.002	-0.005	-0.006	-0.006	-0.002	-0.001	-0.007	0.008	0.008	0.007	0.008	0.007	0.004
		(0.344)	(0.341)	(0.345)	(0.345)	(0.353)	(0.367)	(0.236)	(0.237)	(0.238)	(0.245)	(0.237)	(0.243)	(0.217)	(0.218)	(0.218)	(0.215)	(0.215)	(0.221)
	0.1	0.088	0.090	0.095	0.092	0.095	0.104	0.107	0.107	0.111	0.110	0.114	0.111	0.109	0.110	0.110	0.111	0.110	0.113
		(0.334)	(0.332)	(0.342)	(0.335)	(0.356)	(0.353)	(0.270)	(0.269)	(0.272)	(0.271)	(0.276)	(0.272)	(0.221)	(0.221)	(0.222)	(0.221)	(0.223)	(0.229)
	0.3	0.319	0.323	0.331	0.339	0.341	0.354	0.307	0.309	0.311	0.313	0.318	0.323	0.318	0.318	0.319	0.321	0.320	0.324
		(0.362)	(0.360)	(0.362)	(0.374)	(0.369)	(0.369)	(0.251)	(0.250)	(0.245)	(0.252)	(0.253)	(0.252)	(0.229)	(0.229)	(0.230)	(0.230)	(0.230)	(0.234)
α = 0.5	-0.3	-0.275	-0.290	-0.310	-0.326	-0.346	-0.358	-0.299	-0.305	-0.323	-0.334	-0.339	-0.359	-0.290	-0.295	-0.304	-0.308	-0.322	-0.328
(0,d,0)		(0.227)	(0.232)	(0.230)	(0.245)	(0.246)	(0.259)	(0.174)	(0.174)	(0.180)	(0.175)	(0.182)	(0.189)	(0.131)	(0.132)	(0.131)	(0.133)	(0.135)	(0.134)
	-0.1	-0.101	-0.102	-0.107	-0.124	-0.134	-0.127	-0.111	-0.114	-0.119	-0.116	-0.122	-0.121	-0.095	-0.097	-0.099	-0.104	-0.102	-0.106
		(0.241)	(0.242)	(0.256)	(0.247)	(0.249)	(0.271)	(0.161)	(0.161)	(0.165)	(0.164)	(0.170)	(0.168)	(0.140)	(0.140)	(0.143)	(0.140)	(0.141)	(0.141)
	0	0.001	-0.001	-0.003	-0.001	0.000	-0.003	0.001	0.001	0.000	0.002	0.000	0.004	0.012	0.013	0.013	0.011	0.009	0.014
		(0.240)	(0.238)	(0.244)	(0.244)	(0.248)	(0.271)	(0.163)	(0.165)	(0.165)	(0.170)	(0.170)	(0.172)	(0.137)	(0.137)	(0.132)	(0.137)	(0.136)	(0.135)
	0.1	0.101	0.103	0.114	0.116	0.119	0.134	0.094	0.095	0.095	0.099	0.101	0.106	0.095	0.097	0.097	0.099	0.097	0.103
		(0.227)	(0.223)	(0.223)	(0.237)	(0.238)	(0.243)	(0.179)	(0.178)	(0.180)	(0.181)	(0.182)	(0.185)	(0.141)	(0.141)	(0.142)	(0.139)	(0.144)	(0.146)
	0.3	0.314	0.323	0.331	0.351	0.365	0.389	0.299	0.303	0.311	0.314	0.324	0.335	0.295	0.296	0.297	0.302	0.304	0.311
		(0.225)	(0.224)	(0.235)	(0.232)	(0.239)	(0.252)	(0.167)	(0.167)	(0.167)	(0.171)	(0.169)	(0.169)	(0.133)	(0.133)	(0.133)	(0.133)	(0.134)	(0.133)
α = 0.6	-0.3	-0.272	-0.296	-0.326	-0.343			-0.296	-0.304	-0.335	-0.346	-0.358	-0.379	-0.294	-0.304	-0.317	-0.330	-0.347	-0.360
(0,d,0)		(0.169)	(0.169)	(0.168)	(0.190)			(0.126)	(0.125)	(0.128)	(0.130)	(0.133)	(0.136)	(0.094)	(0.093)	(0.094)	(0.095)	(0.096)	(0.101)
	-0.1	-0.096	-0.102	-0.124	-0.122			-0.085	-0.091	-0.097	-0.103	-0.108	-0.108	-0.107	-0.111	-0.115	-0.118	-0.123	-0.126
		(0.153)	(0.157)	(0.165)	(0.180)			(0.116)	(0.116)	(0.118)	(0.122)	(0.126)	(0.127)	(0.089)	(0.089)	(0.088)	(0.094)	(0.093)	(0.096)
	0	0.001	0.003	0.009	0.002			0.005	0.004	0.002	0.004	0.005	0.008	-0.003	-0.004	-0.003	-0.006	-0.005	-0.002
		(0.155)	(0.159)	(0.164)	(0.177)			(0.117)	(0.117)	(0.120)	(0.122)	(0.122)	(0.127)	(0.090)	(0.091)	(0.093)	(0.094)	(0.092)	(0.094)
	0.1	0.102	0.109	0.119	0.129			0.098	0.103	0.103	0.110	0.114	0.121	0.098	0.101	0.103	0.107	0.112	0.117
		(0.162)	(0.166)	(0.177)	(0.181)			(0.112)	(0.116)	(0.117)	(0.116)	(0.117)	(0.128)	(0.099)	(0.099)	(0.100)	(0.101)	(0.101)	(0.106)
	0.3	0.310	0.329	0.350	0.377			0.312	0.321	0.333	0.349	0.369	0.381	0.300	0.305	0.313	0.323	0.334	0.348
		(0.174)	(0.175)	(0.182)	(0.193)			(0.112)	(0.115)	(0.118)	(0.117)	(0.126)	(0.130)	(0.092)	(0.093)	(0.093)	(0.096)	(0.098)	(0.093)

Table 2: Mean and standard deviation of GSPR estimates from various levels of aggregation, holding fixed the number of periodogram ordinates for ARFIMA(0,d,0). $G(T) = T^{\alpha}$, α = 0.4, 0.5 and 0.6.

	T	200						500						1000					
	d \ n	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
α = 0.4	-0.3	-0,311	-0,323	-0,324	-0,345	-0,354	-0,359	-0,322	-0,327	-0,326	-0,340	-0,344	-0,345	-0,313	-0,315	-0,318	-0,320	-0,325	-0,328
(0,d,0)		(0,301)	(0,304)	(0,291)	(0,312)	(0,312)	(0,299)	(0,214)	(0,214)	(0,213)	(0,217)	(0,221)	(0,210)	(0,184)	(0,182)	(0,188)	(0,183)	(0,180)	(0,190)
	-0.1	-0,133	-0,135	-0,131	-0,139	-0,148	-0,153	-0,123	-0,123	-0,125	-0,125	-0,128	-0,131	-0,114	-0,115	-0,113	-0,117	-0,115	-0,117
		(0,326)	(0,326)	(0,330)	(0,322)	(0,323)	(0,325)	(0,232)	(0,232)	(0,235)	(0,233)	(0,233)	(0,236)	(0,185)	(0,185)	(0,185)	(0,184)	(0,185)	(0,187)
	0	-0,027	-0,027	-0,034	-0,027	-0,029	-0,040	-0,031	-0,031	-0,032	-0,029	-0,027	-0,033	-0,021	-0,021	-0,022	-0,021	-0,020	-0,023
		(0,298)	(0,298)	(0,297)	(0,290)	(0,296)	(0,300)	(0,207)	(0,208)	(0,207)	(0,210)	(0,206)	(0,209)	(0,183)	(0,184)	(0,183)	(0,184)	(0,184)	(0,183)
	0.1	0,053	0,055	0,056	0,057	0,063	0,062	0,083	0,083	0,083	0,085	0,086	0,088	0,080	0,081	0,080	0,081	0,082	0,081
		(0,302)	(0,301)	(0,296)	(0,300)	(0,304)	(0,297)	(0,224)	(0,225)	(0,224)	(0,226)	(0,224)	(0,224)	(0,188)	(0,188)	(0,189)	(0,188)	(0,188)	(0,189)
	0.3	0,279	0,282	0,285	0,290	0,295	0,297	0,277	0,279	0,279	0,282	0,284	0,287	0,285	0,285	0,286	0,287	0,287	0,289
		(0,309)	(0,309)	(0,308)	(0,311)	(0,311)	(0,301)	(0,214)	(0,214)	(0,213)	(0,213)	(0,213)	(0,215)	(0,188)	(0,188)	(0,189)	(0,188)	(0,188)	(0,189)
α = 0.5	-0.3	-0.308	-0.321	-0.332	-0.344	-0.352	-0.351	-0.309	-0.316	-0.326	-0.341	-0.344	-0.351	-0.302	-0.306	-0.312	-0.318	-0.327	-0.333
(0,d,0)		(0.198)	(0.201)	(0.194)	(0.203)	(0.198)	(0.199)	(0.149)	(0.150)	(0.148)	(0.145)	(0.150)	(0.150)	(0.108)	(0.107)	(0.107)	(0.109)	(0.112)	(0.111)
	-0.1	-0.123	-0.124	-0.129	-0.140	-0.143	-0.133	-0.130	-0.131	-0.134	-0.133	-0.135	-0.137	-0.110	-0.112	-0.112	-0.116	-0.114	-0.119
		(0.202)	(0.201)	(0.203)	(0.201)	(0.191)	(0.199)	(0.129)	(0.128)	(0.130)	(0.130)	(0.134)	(0.130)	(0.114)	(0.115)	(0.115)	(0.114)	(0.114)	(0.115)
	0	-0.025	-0.024	-0.027	-0.024	-0.025	-0.025	-0.011	-0.011	-0.012	-0.013	-0.009	-0.013	-0.002	-0.002	-0.002	-0.003	-0.004	0.000
		(0.212)	(0.210)	(0.209)	(0.208)	(0.204)	(0.212)	(0.138)	(0.139)	(0.138)	(0.140)	(0.137)	(0.140)	(0.110)	(0.110)	(0.109)	(0.109)	(0.110)	(0.109)
	0.1	0.083	0.085	0.087	0.091	0.093	0.096	0.081	0.082	0.084	0.084	0.087	0.090	0.083	0.084	0.085	0.087	0.086	0.089
		(0.193)	(0.194)	(0.188)	(0.197)	(0.186)	(0.186)	(0.144)	(0.144)	(0.144)	(0.144)	(0.143)	(0.145)	(0.110)	(0.111)	(0.111)	(0.110)	(0.112)	(0.112)
	0.3	0.295	0.303	0.308	0.317	0.325	0.337	0.291	0.294	0.297	0.301	0.306	0.313	0.286	0.287	0.289	0.291	0.293	0.297
		(0.186)	(0.187)	(0.185)	(0.187)	(0.185)	(0.192)	(0.132)	(0.132)	(0.133)	(0.134)	(0.133)	(0.133)	(0.112)	(0.112)	(0.112)	(0.112)	(0.112)	(0.111)
α = 0.6	-0.3	-0,298	-0,314	-0,324	-0,328			-0,301	-0,310	-0,327	-0,336	-0,341	-0,336	-0,302	-0,310	-0,322	-0,329	-0,339	-0,343
(0,d,0)		(0,137)	(0,140)	(0,136)	(0,135)			(0,101)	(0,101)	(0,097)	(0,097)	(0,100)	(0,097)	(0,073)	(0,073)	(0,073)	(0,073)	(0,074)	(0,076)
	-0.1	-0,113	-0,117	-0,127	-0,122			-0,095	-0,098	-0,101	-0,107	-0,109	-0,107	-0,109	-0,112	-0,114	-0,117	-0,118	-0,121
		(0,126)	(0,127)	(0,125)	(0,133)			(0,093)	(0,093)	(0,093)	(0,095)	(0,093)	(0,091)	(0,070)	(0,070)	(0,070)	(0,070)	(0,072)	(0,071)
	0	-0,007	-0,006	-0,006	-0,006			-0,011	-0,011	-0,011	-0,013	-0,008	-0,008	-0,009	-0,008	-0,009	-0,009	-0,009	-0,009
		(0,130)	(0,128)	(0,130)	(0,128)			(0,094)	(0,094)	(0,093)	(0,095)	(0,090)	(0,091)	(0,071)	(0,071)	(0,072)	(0,071)	(0,070)	(0,071)
	0.1	0,086	0,089	0,097	0,097			0,084	0,087	0,089	0,094	0,095	0,099	0,092	0,094	0,096	0,098	0,101	0,105
		(0,136)	(0,134)	(0,136)	(0,134)			(0,095)	(0,095)	(0,096)	(0,095)	(0,091)	(0,096)	(0,077)	(0,076)	(0,077)	(0,076)	(0,078)	(0,075)
	0.3	0,293	0,304	0,315	0,317			0,299	0,304	0,313	0,320	0,328	0,325	0,297	0,300	0,306	0,312	0,317	0,324
		(0,142)	(0,141)	(0,139)	(0,139)			(0,096)	(0,095)	(0,096)	(0,095)	(0,098)	(0,098)	(0,073)	(0,073)	(0,073)	(0,073)	(0,074)	(0,073)

Table 3: Mean and standard deviation of GPH estimates from various levels of aggregation, holding fixed the number of periodogram ordinates for ARFIMA(1,d,0), ϕ = 0.8, and ARFIMA(0,d,1), θ = -0.8. $G(T) = T^{\alpha}$, α = 0.5.

	T	200						500						1000					
GPH	d \ n	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
α = 0.5	-0.3	0.019	0.025	0.040	0.042	0.055	0.064	-0.145	-0.143	-0.139	-0.135	-0.128	-0.124	-0.204	-0.203	-0.201	-0.199	-0.197	-0.193
(1,d,0)		(0.235)	(0.234)	(0.236)	(0.246)	(0.253)	(0.249)	(0.168)	(0.169)	(0.170)	(0.170)	(0.173)	(0.174)	(0.149)	(0.149)	(0.151)	(0.150)	(0.151)	(0.154)
	-0.1	0.204	0.212	0.232	0.245	0.261	0.285	0.053	0.056	0.061	0.066	0.075	0.085	-0.020	-0.019	-0.016	-0.014	-0.011	-0.006
		(0.226)	(0.229)	(0.231)	(0.236)	(0.252)	(0.254)	(0.174)	(0.175)	(0.176)	(0.177)	(0.179)	(0.181)	(0.137)	(0.137)	(0.137)	(0.137)	(0.137)	(0.138)
	0	0.288	0.299	0.318	0.339	0.355	0.386	0.161	0.164	0.169	0.176	0.187	0.196	0.086	0.088	0.091	0.094	0.097	0.104
		(0.225)	(0.229)	(0.235)	(0.241)	(0.236)	(0.265)	(0.173)	(0.173)	(0.174)	(0.175)	(0.177)	(0.178)	(0.134)	(0.135)	(0.135)	(0.136)	(0.135)	(0.137)
	0.1	0.417	0.428	0.444	0.469	0.500	0.519	0.273	0.276	0.282	0.291	0.302	0.313	0.194	0.195	0.198	0.201	0.207	0.214
		(0.210)	(0.211)	(0.216)	(0.219)	(0.230)	(0.238)	(0.172)	(0.172)	(0.174)	(0.175)	(0.178)	(0.178)	(0.136)	(0.137)	(0.135)	(0.137)	(0.138)	(0.136)
	0.3	0.612	0.623	0.651	0.673	0.705	0.742	0.469	0.473	0.480	0.489	0.503	0.517	0.390	0.392	0.395	0.399	0.404	0.411
		(0.223)	(0.224)	(0.230)	(0.230)	(0.238)	(0.249)	(0.178)	(0.178)	(0.181)	(0.181)	(0.182)	(0.186)	(0.147)	(0.147)	(0.147)	(0.148)	(0.148)	(0.149)
α = 0.5	-0.3	-0.455	-0.653	-0.718	-0.758	-0.771	-0.789	-0.350	-0.548	-0.628	-0.665	-0.696	-0.723	-0.321	-0.489	-0.562	-0.612	-0.642	-0.666
(0,d,1)		(0.249)	(0.262)	(0.260)	(0.258)	(0.266)	(0.282)	(0.192)	(0.195)	(0.202)	(0.197)	(0.194)	(0.204)	(0.156)	(0.163)	(0.158)	(0.160)	(0.162)	(0.160)
	-0.1	-0.364	-0.504	-0.566	-0.614	-0.655	-0.658	-0.243	-0.362	-0.422	-0.474	-0.500	-0.536	-0.170	-0.260	-0.318	-0.362	-0.396	-0.428
		(0.250)	(0.249)	(0.246)	(0.254)	(0.256)	(0.254)	(0.185)	(0.181)	(0.168)	(0.191)	(0.178)	(0.191)	(0.135)	(0.137)	(0.134)	(0.143)	(0.139)	(0.137)
	0	-0.277	-0.393	-0.479	-0.502	-0.536	-0.556	-0.140	-0.225	-0.293	-0.324	-0.357	-0.384	-0.091	-0.143	-0.195	-0.229	-0.267	-0.281
		(0.229)	(0.225)	(0.249)	(0.244)	(0.253)	(0.281)	(0.176)	(0.180)	(0.172)	(0.181)	(0.183)	(0.180)	(0.134)	(0.135)	(0.137)	(0.134)	(0.133)	(0.137)
	0.1	-0.198	-0.286	-0.334	-0.381	-0.391	-0.415	-0.049	-0.113	-0.160	-0.198	-0.214	-0.250	0.009	-0.030	-0.068	-0.090	-0.111	-0.142
		(0.253)	(0.233)	(0.233)	(0.256)	(0.254)	(0.240)	(0.173)	(0.170)	(0.164)	(0.166)	(0.171)	(0.170)	(0.135)	(0.134)	(0.133)	(0.140)	(0.138)	(0.136)
	0.3	-0.006	-0.056	-0.097	-0.124	-0.135	-0.150	0.152	0.126	0.094	0.076	0.070	0.046	0.217	0.203	0.188	0.173	0.164	0.154
		(0.242)	(0.243)	(0.258)	(0.255)	(0.258)	(0.276)	(0.183)	(0.186)	(0.176)	(0.185)	(0.181)	(0.186)	(0.143)	(0.142)	(0.139)	(0.144)	(0.149)	(0.144)

Table 4: Mean and standard deviation of GSPR estimates from various levels of aggregation, holding fixed the number of periodogram ordinates for ARFIMA(1,d,0), $\phi = 0.8$, and ARFIMA(0,d,1), $\theta = -0.8$. $G(T) = T^{\alpha}$, $\alpha = 0.5$.

	T	200						500						1000					
GPH	d \ n	1	2	3	4	5	6	1	2	3	4	5	6	1	2	3	4	5	6
$\alpha = 0.5$	-0.3	0,003	0,012	0,025	0,039	0,050	0,063	-0,148	-0,145	-0,140	-0,133	-0,127	-0,118	-0,212	-0,210	-0,208	-0,205	-0,202	-0,197
(1,d,0)		(0,196)	(0,196)	(0,196)	(0,200)	(0,200)	(0,197)	(0,135)	(0,135)	(0,136)	(0,136)	(0,135)	(0,136)	(0,121)	(0,121)	(0,121)	(0,121)	(0,121)	(0,122)
	-0.1	0,197	0,206	0,225	0,236	0,251	0,267	0,043	0,046	0,051	0,057	0,065	0,074	-0,030	-0,029	-0,026	-0,023	-0,020	-0,015
		(0,205)	(0,206)	(0,208)	(0,206)	(0,205)	(0,210)	(0,145)	(0,145)	(0,145)	(0,145)	(0,145)	(0,146)	(0,109)	(0,109)	(0,109)	(0,109)	(0,109)	(0,109)
	0	0,280	0,288	0,305	0,320	0,334	0,356	0,149	0,152	0,158	0,164	0,173	0,183	0,079	0,080	0,083	0,086	0,090	0,096
		(0,198)	(0,198)	(0,198)	(0,200)	(0,196)	(0,205)	(0,151)	(0,151)	(0,151)	(0,151)	(0,152)	(0,151)	(0,113)	(0,113)	(0,113)	(0,113)	(0,113)	(0,113)
	0.1	0,402	0,411	0,427	0,440	0,462	0,473	0,261	0,264	0,270	0,276	0,285	0,295	0,179	0,180	0,182	0,186	0,190	0,195
		(0,189)	(0,190)	(0,190)	(0,191)	(0,192)	(0,194)	(0,142)	(0,142)	(0,142)	(0,143)	(0,144)	(0,142)	(0,108)	(0,108)	(0,109)	(0,108)	(0,108)	(0,110)
	0.3	0,600	0,608	0,625	0,638	0,656	0,673	0,455	0,458	0,464	0,470	0,479	0,489	0,383	0,384	0,387	0,390	0,394	0,400
		(0,196)	(0,196)	(0,196)	(0,198)	(0,200)	(0,197)	(0,145)	(0,145)	(0,145)	(0,145)	(0,146)	(0,145)	(0,115)	(0,115)	(0,115)	(0,115)	(0,115)	(0,115)
$\alpha = 0.5$	-0.3	-0,495	-0,673	-0,732	-0,754	-0,749	-0,735	-0,379	-0,567	-0,640	-0,672	-0,695	-0,718	-0,340	-0,502	-0,570	-0,615	-0,645	-0,667
(0,d,1)		(0,213)	(0,228)	(0,216)	(0,214)	(0,210)	(0,213)	(0,161)	(0,165)	(0,169)	(0,163)	(0,157)	(0,166)	(0,126)	(0,133)	(0,133)	(0,137)	(0,137)	(0,132)
	-0.1	-0,398	-0,524	-0,585	-0,619	-0,630	-0,624	-0,258	-0,376	-0,433	-0,481	-0,505	-0,525	-0,189	-0,277	-0,331	-0,375	-0,404	-0,427
		(0,214)	(0,212)	(0,210)	(0,207)	(0,205)	(0,208)	(0,154)	(0,154)	(0,144)	(0,161)	(0,144)	(0,151)	(0,111)	(0,115)	(0,112)	(0,117)	(0,114)	(0,114)
	0	-0,312	-0,423	-0,493	-0,508	-0,524	-0,523	-0,167	-0,246	-0,308	-0,339	-0,369	-0,389	-0,104	-0,159	-0,207	-0,240	-0,272	-0,287
		(0,205)	(0,195)	(0,208)	(0,201)	(0,201)	(0,217)	(0,147)	(0,154)	(0,146)	(0,150)	(0,149)	(0,147)	(0,114)	(0,115)	(0,117)	(0,115)	(0,117)	(0,115)
	0.1	-0,221	-0,306	-0,356	-0,389	-0,397	-0,406	-0,069	-0,132	-0,178	-0,210	-0,232	-0,263	-0,005	-0,044	-0,078	-0,104	-0,126	-0,150
		(0,206)	(0,204)	(0,191)	(0,208)	(0,206)	(0,193)	(0,131)	(0,136)	(0,139)	(0,140)	(0,145)	(0,143)	(0,115)	(0,114)	(0,113)	(0,118)	(0,119)	(0,117)
	0.3	-0,033	-0,085	-0,124	-0,148	-0,162	-0,173	0,134	0,107	0,078	0,060	0,048	0,027	0,199	0,184	0,171	0,153	0,145	0,135
		(0,209)	(0,207)	(0,216)	(0,208)	(0,207)	(0,217)	(0,145)	(0,149)	(0,140)	(0,146)	(0,145)	(0,146)	(0,117)	(0,119)	(0,115)	(0,118)	(0,121)	(0,114)

Table 5: Correlation between GPH estimates from the original series and from the aggregated series; level of aggregation (n) up to 6, holding fixed the number of periodogram ordinates used in the estimation. a) ARFIMA(0,d,0); $F(T) = T^{0.4}$. b) ARFIMA(0,d,0); $F(T) = T^{0.5}$. c) ARFIMA(0,d,0); $F(T) = T^{0.6}$. d) ARFIMA(1,d,0); $\phi = 0.8$; $F(T) = T^{0.5}$. e) ARFIMA(0,d,1); $\theta = -0.8$; $F(T) = T^{0.5}$.

	T	200					500					1000				
	d n	2	3	4	5	6	2	3	4	5	6	2	3	4	5	6
	-0.3	0.949	0.803	0.849	0.824	0.690	0.973	0.870	0.917	0.888	0.806	0.987	0.930	0.957	0.930	0.858
	-0.1	0.978	0.903	0.922	0.904	0.849	0.990	0.947	0.962	0.950	0.925	0.996	0.982	0.982	0.971	0.944
a)	0	0.985	0.923	0.949	0.929	0.864	0.994	0.968	0.970	0.967	0.931	0.997	0.988	0.990	0.988	0.958
	0.1	0.991	0.939	0.957	0.942	0.899	0.997	0.976	0.988	0.979	0.963	0.998	0.994	0.993	0.992	0.974
	0.3	0.994	0.960	0.971	0.965	0.929	0.998	0.985	0.994	0.987	0.973	1.000	0.997	0.997	0.996	0.982
	-0.3	0.931	0.765	0.802	0.757	0.630	0.953	0.851	0.866	0.863	0.755	0.966	0.908	0.902	0.884	0.790
	-0.1	0.947	0.856	0.879	0.840	0.750	0.977	0.928	0.917	0.883	0.856	0.991	0.966	0.960	0.945	0.900
b)	0	0.964	0.899	0.900	0.862	0.802	0.984	0.944	0.947	0.908	0.881	0.990	0.973	0.970	0.955	0.908
	0.1	0.972	0.906	0.908	0.875	0.823	0.989	0.959	0.964	0.946	0.902	0.994	0.985	0.980	0.968	0.939
	0.3	0.983	0.927	0.925	0.888	0.840	0.995	0.973	0.980	0.968	0.929	0.997	0.992	0.990	0.981	0.954
	-0.3	0.886	0.735	0.752			0.939	0.825	0.832	0.800	0.701	0.948	0.897	0.869	0.840	0.758
	-0.1	0.914	0.799	0.802			0.954	0.873	0.865	0.853	0.773	0.969	0.923	0.908	0.872	0.824
c)	0	0.924	0.812	0.792			0.965	0.909	0.891	0.845	0.815	0.975	0.943	0.915	0.893	0.816
	0.1	0.941	0.839	0.822			0.969	0.908	0.895	0.848	0.820	0.985	0.962	0.949	0.927	0.882
	0.3	0.965	0.894	0.860			0.982	0.935	0.926	0.874	0.848	0.989	0.973	0.960	0.934	0.884
	-0.3	0.993	0.937	0.957	0.932	0.859	0.998	0.969	0.987	0.978	0.944	0.999	0.994	0.995	0.991	0.955
	-0.1	0.997	0.951	0.973	0.943	0.894	0.999	0.982	0.994	0.989	0.967	0.999	0.994	0.996	0.994	0.964
d)	0	0.997	0.953	0.978	0.962	0.907	0.999	0.980	0.995	0.990	0.969	1.000	0.996	0.999	0.997	0.966
	0.1	0.997	0.950	0.978	0.949	0.869	1.000	0.977	0.996	0.993	0.966	1.000	0.997	0.998	0.998	0.966
	0.3	0.997	0.946	0.979	0.959	0.883	1.000	0.983	0.998	0.995	0.976	1.000	0.996	0.999	0.998	0.971
	-0.3	0.688	0.208	0.462	0.494	0.157	0.742	0.231	0.620	0.558	0.185	0.741	0.290	0.663	0.630	0.292
	-0.1	0.738	0.361	0.582	0.521	0.383	0.813	0.573	0.650	0.627	0.475	0.825	0.641	0.729	0.708	0.529
e)	0	0.747	0.490	0.634	0.541	0.453	0.817	0.677	0.712	0.684	0.596	0.863	0.733	0.772	0.718	0.643
	0.1	0.807	0.584	0.684	0.623	0.528	0.843	0.702	0.706	0.742	0.605	0.893	0.782	0.792	0.779	0.731
	0.3	0.877	0.780	0.784	0.761	0.742	0.925	0.858	0.842	0.817	0.767	0.952	0.921	0.901	0.873	0.835

Table 6: Correlation between GSPR estimates from the original series and from the aggregated series; level of aggregation (n) up to 6, holding fixed the number of periodogram ordinates used in the estimation. a) ARFIMA(0,d,0); $F(T) = T^{0.4}$. b) ARFIMA(0,d,0); $F(T) = T^{0.5}$. c) ARFIMA(0,d,0); $F(T) = T^{0.6}$. d) ARFIMA(1,d,0); $\phi = 0.8$; $F(T) = T^{0.5}$. e) ARFIMA(0,d,1); $\theta = -0.8$; $F(T) = T^{0.5}$.

	T	200					500					1000				
	d n	2	3	4	5	6	2	3	4	5	6	2	3	4	5	6
	-0.3	0,976	0,873	0,911	0,879	0,785	0,990	0,934	0,960	0,941	0,887	0,995	0,974	0,984	0,971	0,913
	-0.1	0,991	0,951	0,961	0,943	0,896	0,997	0,982	0,986	0,979	0,965	0,999	0,995	0,995	0,991	0,981
a)	0	0,994	0,970	0,971	0,956	0,919	0,998	0,988	0,991	0,983	0,968	0,999	0,997	0,997	0,995	0,987
	0.1	0,996	0,974	0,981	0,967	0,939	0,999	0,994	0,995	0,992	0,984	1,000	0,998	0,999	0,997	0,992
	0.3	0,999	0,987	0,991	0,986	0,968	1,000	0,997	0,998	0,996	0,993	1,000	1,000	0,999	0,999	0,996
	-0.3	0.963	0.852	0.880	0.829	0.746	0.985	0.933	0.929	0.908	0.853	0.988	0.952	0.953	0.942	0.874
	-0.1	0.982	0.928	0.933	0.886	0.827	0.991	0.964	0.962	0.941	0.909	0.997	0.988	0.983	0.974	0.956
b)	0	0.988	0.953	0.943	0.918	0.884	0.995	0.979	0.974	0.958	0.938	0.997	0.992	0.990	0.979	0.966
	0.1	0.988	0.962	0.946	0.922	0.868	0.997	0.986	0.983	0.973	0.951	0.999	0.995	0.992	0.986	0.977
	0.3	0.994	0.970	0.964	0.942	0.882	0.999	0.992	0.991	0.986	0.971	1.000	0.998	0.997	0.995	0.989
	-0.3	0,941	0,848	0,834			0,968	0,900	0,885	0,844	0,774	0,975	0,941	0,914	0,877	0,828
	-0.1	0,954	0,884	0,848			0,977	0,940	0,905	0,900	0,834	0,988	0,967	0,943	0,913	0,894
c)	0	0,962	0,890	0,833			0,986	0,953	0,930	0,890	0,844	0,990	0,978	0,958	0,941	0,905
	0.1	0,972	0,914	0,869			0,990	0,972	0,942	0,906	0,879	0,995	0,983	0,975	0,957	0,929
	0.3	0,984	0,941	0,884			0,994	0,978	0,959	0,919	0,894	0,997	0,991	0,983	0,963	0,939
	-0.3	0,998	0,983	0,981	0,967	0,908	0,999	0,993	0,995	0,990	0,977	1,000	0,999	0,999	0,997	0,988
	-0.1	0,999	0,987	0,989	0,972	0,944	1,000	0,997	0,998	0,996	0,989	1,000	0,999	0,999	0,998	0,991
d)	0	0,999	0,987	0,990	0,977	0,939	1,000	0,997	0,999	0,997	0,991	1,000	1,000	1,000	0,999	0,995
	0.1	0,999	0,987	0,991	0,976	0,936	1,000	0,997	0,999	0,998	0,992	1,000	1,000	1,000	0,999	0,996
	0.3	1,000	0,988	0,994	0,983	0,947	1,000	0,997	0,999	0,998	0,993	1,000	1,000	1,000	1,000	0,996
	-0.3	0,738	0,248	0,533	0,571	0,157	0,800	0,257	0,695	0,632	0,210	0,832	0,386	0,754	0,741	0,361
	-0.1	0,814	0,459	0,652	0,607	0,407	0,861	0,632	0,755	0,701	0,564	0,881	0,731	0,801	0,766	0,648
e)	0	0,837	0,650	0,734	0,650	0,520	0,888	0,774	0,808	0,753	0,694	0,920	0,838	0,854	0,800	0,763
	0.1	0,869	0,678	0,757	0,712	0,628	0,890	0,799	0,781	0,781	0,694	0,943	0,886	0,876	0,866	0,817
	0.3	0,933	0,855	0,841	0,816	0,780	0,960	0,921	0,891	0,866	0,844	0,979	0,952	0,941	0,924	0,894

Table 7: GPH and GSPR estimates from aggregates; level of aggregation up to 10 (n = 1 for the original series), holding fixed the number of periodogram ordinates used in the estimation (m = 10, 20, 30, 40, 60, 80, 100, 120, 150). Daily US$/FF exchange rate series from October 20, 1977 to October 23, 2002 (25 years); log of the squared returns.

GPH

n\m	10	20	30	40	60	80	100	120	150
1	-0,049	0,246	0,297	0,250	0,266	0,309	0,343	0,322	0,284
2	-0,049	0,245	0,297	0,251	0,267	0,306	0,341	0,320	0,286
3	-0,051	0,246	0,296	0,249	0,269	0,308	0,342	0,320	0,277
4	-0,048	0,244	0,299	0,250	0,266	0,310	0,350	0,327	0,284
5	-0,049	0,245	0,293	0,251	0,267	0,308	0,343	0,323	0,289
6	-0,054	0,242	0,288	0,251	0,290	0,307	0,337	0,316	0,282
7	-0,050	0,242	0,292	0,255	0,263	0,294	0,348	0,339	0,288
8	-0,053	0,242	0,291	0,263	0,279	0,312	0,371	0,345	0,294
9	-0,054	0,242	0,288	0,255	0,264	0,300	0,343	0,355	0,306
10	-0,057	0,246	0,287	0,268	0,277	0,296	0,334	0,323	0,312

GSPR

n\m	10	20	30	40	60	80	100	120	150
1	0,245	0,456	0,379	0,277	0,309	0,339	0,356	0,350	0,292
2	0,245	0,456	0,378	0,276	0,310	0,338	0,356	0,348	0,290
3	0,243	0,457	0,379	0,276	0,309	0,338	0,356	0,349	0,290
4	0,246	0,454	0,379	0,275	0,309	0,339	0,360	0,350	0,286
5	0,245	0,456	0,377	0,277	0,307	0,336	0,346	0,341	0,300
6	0,243	0,456	0,377	0,280	0,309	0,337	0,358	0,353	0,297
7	0,245	0,455	0,380	0,278	0,313	0,340	0,360	0,358	0,295
8	0,243	0,459	0,377	0,275	0,308	0,337	0,369	0,350	0,299
9	0,242	0,460	0,373	0,278	0,307	0,340	0,365	0,365	0,295
10	0,243	0,466	0,377	0,280	0,307	0,328	0,345	0,335	0,306

Figure 1: US$/FF exchange rate, logarithm of the squared returns from October 20, 1977 to October 23, 2002. a) The series Z_t; b) ACF of Z_t; c) Periodogram of Z_t in log-log scale.